Eigenresidual Tolerances for Spectral Partitioning

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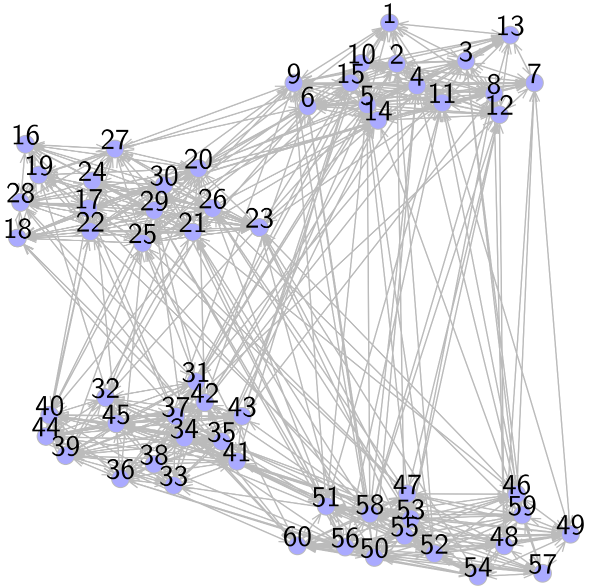
Graph Partitioning



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Spectral Graph Theory

Graphs and Linear Algebra	
Adjacency Matrix	$A_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j \\ 0, & \text{otherwise} \end{cases}$
Degrees	$D_{ii} = d_i = \deg(v_i)$ all other entries 0.
Combinatorial Laplacian	$L = D - A$
Normalized Laplacian	$\hat{L} = I - D^{-1/2} A D^{-1/2}$
Normalized Adjacency	$\hat{A} = D^{-1/2} A D^{-1/2}$

Defining Accuracy for the Numerical Problem [Golub and Van Loan, 2013]

Eigenpair $\lambda_i, \mathbf{v}_i$	$\hat{L}\mathbf{v}_i = \lambda_i \mathbf{v}_i$
Symmetric Positive Semidefinite	$\lambda_1 = 0 \leq \lambda_2 \leq \ldots \lambda_n \leq 2 \in \mathbb{R}^+$
Rayleigh Quotient	$\mu = \frac{\mathbf{x}^T \hat{L} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$
Residual	$r = \left\Vert \hat{L}\mathbf{x} - \mu \mathbf{x} \right\Vert$
Error	$\mathbf{z} = \Vert \mathbf{x} - \mathbf{v} \Vert$ where $\hat{L}\mathbf{v} = \lambda \mathbf{v}$

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Define the Quality of Partitioning as Conductance [Chung, 1997]

- Define $E(S) = \sum_{i \in S j \notin S} a_{i,j}$ total weight of edges leaving S.
- Define $\mathrm{Vol}\,(S) = {e_S}^T A e_S$ as total weight of edges within S
- The conductance of a cut S is:

$$\phi\,(S) = \frac{E(S)}{\min\{\mathrm{Vol}\,(S), \mathrm{Vol}\left(\bar{S}\right)\}}$$

- When applying spectral methods, accurate eigenpairs are not the direct goal

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Spectral Sweep Cut Algorithm

- Compute $\mathbf{x}$ so that $\left\|\hat{L}\mathbf{x} - \mu\mathbf{x}\right\|_2 < \epsilon$
- Sort $D^{-\frac{1}{2}}\mathbf{x}$
- Check all n sweep cuts of $D^{-\frac{1}{2}}\mathbf{x}$ for minimum ϕ

- You need to choose ϵ small enough to get correct cut.
- We provide the first principled method for choosing ϵ for spectral partitioning

General Cheeger's Inequality [Mihail, 1989]

Theorem

If $\mathbf{x}$ is a vector orthogonal to $D^{\frac{1}{2}}\mathbf{1}$ such that $\mathbf{x}^T\hat{L}\mathbf{x} = \mu$, then $\mathbf{y} = D^{-\frac{1}{2}}\mathbf{x}$ has a sweep cut S such that $\phi(S) = \phi\left(D^{-\frac{1}{2}}\mathbf{x}\right) \leq \sqrt{2\mu}$.

- Smallest possible μ is λ_2 so approximate $\mathbf{v}_2$
- Low energy vectors wrt Laplacian yield good partitions
- Goal: show that these vectors are easier to compute with high accuracy

Eigenspace error bounds

We generalize an eigenspace error bound to invariant subspaces.

	[Parlett, 1998]	[F, Sanders 2015]
Target	$\mathbf{x} \approx \mathbf{v}_2$	$\mathbf{x} \approx \sum_2^q \alpha_i \mathbf{v}_i$
Error	$\lVert \mathbf{x} - \mathbf{P}_2 \mathbf{x} \rVert$	$\lVert \mathbf{x} - \mathbf{P}_2^q \mathbf{x} \rVert$
Gap δ	$\lvert \lambda_2 - \lambda_3 \rvert$	$\lvert \mu - \lambda_{q+1} \rvert$
Bound	$\lVert \mathbf{z} \rVert \, \delta_2 < \sqrt{8} \, \lVert \mathbf{r} \rVert$	$\lVert \mathbf{z} \rVert \, \delta_q < \sqrt{2} \, \lVert \mathbf{r} \rVert$
Condition	$\mu \leq \frac{\lambda_2 + \lambda_3}{2}$	$\mu \leq \lambda_q$

Our improvement is based on $\lvert \mu - \lambda_{q+1} \rvert > \lvert \lambda_2 - \lambda_3 \rvert$.

Blends and Blendspaces

- Analyze the problem to determine a set of "good" eigenvectors
- Transition from computing an approximation of a single eigenvector to an approximation of a linear combination
- Give up control over which linear combination you compute
- Gain tighter bounds and faster convergence

Ring of Cliques

$\mathcal{R}_{b,q}$: q blocks each of size b.

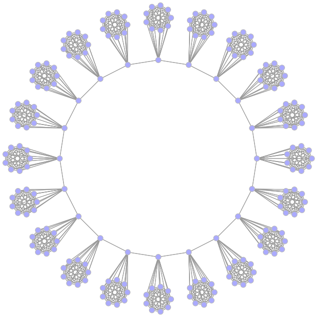

Figure 1: A drawing of $\mathcal{R}_{b,q}$ laid out to show structure.

$$\begin{bmatrix} J_b & \mathbf{e}_1\mathbf{e}_1^t & 0 & \cdots & & \mathbf{e}_1\mathbf{e}_1^t \\ \mathbf{e}_1\mathbf{e}_1^t & J_b & \mathbf{e}_1\mathbf{e}_1^t & 0 & & \\ 0 & \mathbf{e}_1\mathbf{e}_1^t & J_b & \mathbf{e}_1\mathbf{e}_1^t & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & 0 & \mathbf{e}_1\mathbf{e}_1^t & J_b & \mathbf{e}_1\mathbf{e}_1^t \\ \mathbf{e}_1\mathbf{e}_1^t & & & 0 & \mathbf{e}_1\mathbf{e}_1^t & J_b \end{bmatrix}$$

where $\quad J_b = \mathbf{1}_b\mathbf{1}_b^t - I_b.$

Figure 2: The adjacency matrix of $\mathcal{R}_{b,q}$ has block circulant structure.

Ring of Cliques Correct Partitions

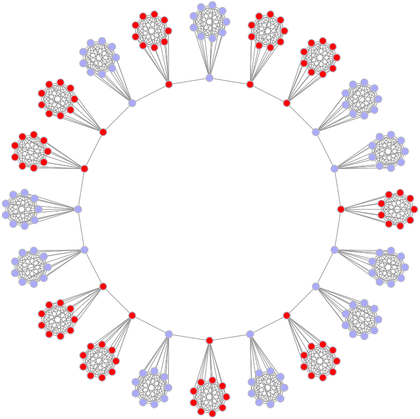

Figure 3: In a "correct" partition of $\mathcal{R}_{b,q}$ all cliques are entirely red or entirely blue.

$\mathcal{R}_{b,q}$ Spectrum

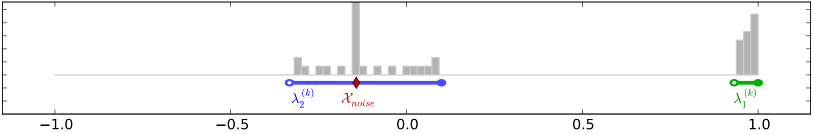

Figure 4: Spectrum of $\hat{A} = I - \hat{L}$ of $\mathcal{R}_{34,10}$.

- *Blend interval* $[\lambda_q, \lambda_p]$ in green
- *Blend space Span*$\{v_2 \dots v_q\}$
- $\lambda(\hat{A}) > \frac{1}{2}$ eigenvectors recover partition
- *Spectral gap* $\delta_2 = \mathcal{O}\left(n^{-2}\right)$
- *Blend gap* $\delta_q = \mathcal{O}\left(1\right)$

Eigenpairs of $\mathcal{R}_{b,q}$ normalized Adjacency Matrix

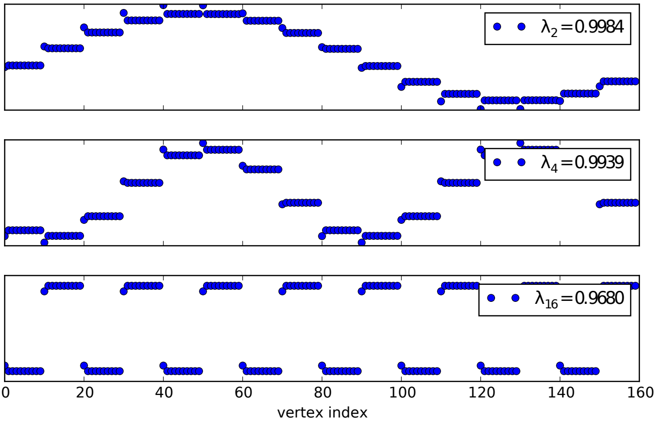

Figure 5: The eigenvectors of $\hat{A}$ are shown for $\mathcal{R}_{10,16}$. The eigenvectors with eigenvalues close to 1 indicate the block structure with differing frequencies.

Constructing a minimal perturbation **z**

Let $\alpha_i = \max_{j \in \mathcal{B}_i} z_i$ and $\beta_i = \min_{j \in \mathcal{B}_i} z_i$ for each $\mathcal{B}_i$.
If for all i $\alpha_i - \beta_{i+1} > \mu_{i+1} - \mu i$ then there is no good sweep cut.

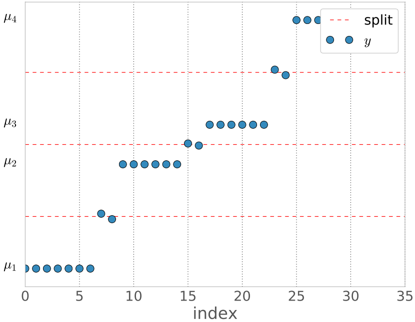

Figure 6: Example of the minimal perturbation construction.

$\mathcal{R}_{b,q}$ Sufficient Errors

Theorem

Let $\mathbf{P}_{\mathcal{W}}$ be the orthogonal projector onto $\mathcal{W}$.
For any vector $\mathbf{y}$, define $\mathbf{z} = (\mathbf{I} - \mathbf{P}_{\mathcal{W}})\mathbf{y}$.
If $\|\mathbf{z}\|_2 \leq (1 + 2qn)^{-\frac{1}{2}} \|\mathbf{y}\|_2$, then $\mathbf{y}$ recovers all the cliques of $\mathcal{R}_{b,q}$.

Sketch of proof.

- Let $\alpha_i = \max_{j \in \mathcal{B}_i} z_i$ and $\beta_i = \min_{j \in \mathcal{B}_i} z_i$ for each $\mathcal{B}_i$.
- Suppose that $\mathbf{y}$ does not recover all the cliques, then for all $\mathcal{B}_i$

$$\alpha_i - \beta_{i+1} > \mu_{i+1} - \mu_i.$$

- Thus we can bound the 1-norm error below as follows:

$$\|\mathbf{z}\|_1 \geq \sum_i (\alpha_i - \beta_{i+1}) \geq \mu_q - \mu_1 \geq n^{-\frac{1}{2}} \|\mathbf{x}\|_2 .$$

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$\mathcal{R}_{b,q}$ Necessary Accuracy

Theorem

For any unit vector $\mathbf{x} \in \mathcal{W}$ *there exists a perturbation* $\mathbf{z}$ *where* $\|\mathbf{z}\| < b^{-\frac{1}{2}}, \mathbf{P}_{\mathcal{W}}\mathbf{z} = 0$ *such that* $\mathbf{y} = \mathbf{x} + \mathbf{z}$ *does not recover all the cliques.*

Sketch of proof.

$$\|\mathbf{z}\|_2^2 = \sum_{i=0}^{q-1} \alpha_i^2 + \beta_{i+1}^2 = \frac{1}{2}\sum_{i=0}^{q-1}(\mu_{i+1} - \mu_i)^2 < b^{-1}\|\mathbf{x}\|_2^2$$

□

$\mathcal{R}_{b,q}$ Minimal Perturbations

Residual tolerances are larger for blends because:

$\delta_2 = \mathcal{O}\left(n^{-2}\right)$ but $\delta_q = \mathcal{O}\left(1\right)$

q	b	sufficient	necessary
$n/25$	25	$\frac{5}{2}n^{-1}$	$\frac{1}{\sqrt{5}}$
25	$n/25$	$\frac{1}{14}n^{-\frac{1}{2}}$	$5n^{-\frac{1}{2}}$
$\sqrt{n}$	$\sqrt{n}$	$\frac{1}{2}n^{-3/4}$	$\frac{1}{2}n^{-1/4}$

Table 1: Error bounds for various parameter regimes.

$\mathcal{R}_{b,q}$ Minimal Perturbations

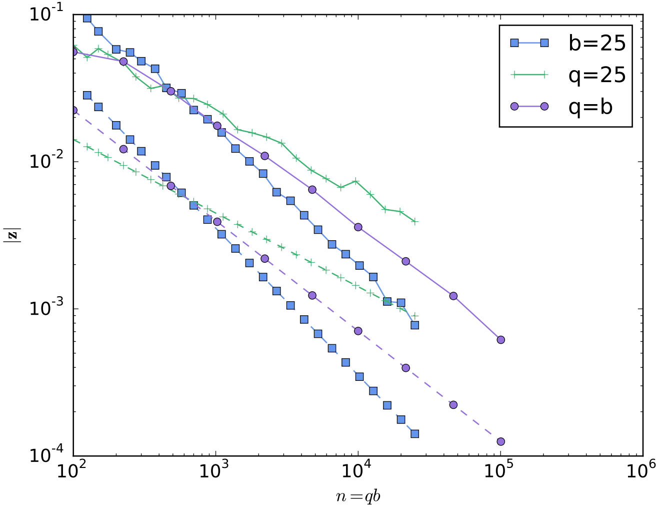

Figure 7: Empirical measurements of minimal error perturbations. Dashed lines indicate lower bounds.

$\mathcal{R}_{b,q}$ Power Method

Theorem

Let $\mathbf{x}_0$ be sampled from $\mathcal{N}_n(0,1)$, with high probability:
The power method $\mathbf{x}_{k+1} \longleftarrow \hat{A}\mathbf{x}_k / \|\mathbf{x}_k\|$ for $\mathcal{R}_{b,q}$.
There is a k^* of $\mathcal{O}(\log_b q)$ such that for $k \geq k^*$ a sweep cut of $\mathbf{x}_k$ recovers the clusters.

For $q \in \mathcal{O}(b^p)$, the number of iterations is $\mathcal{O}(p)$.

Conclusions

- Spectral blends lead to lower accuracy requirements in spectral partitioning
- Residual tolerance necessary for Fiedler partitions is $\mathcal{O}\left(n^{-5/2}\right)$ while data clusters are resolved at $\mathcal{O}\left(n^{-1/2}\right)$ for $\mathcal{R}_{b,q}$
- Analyzing model problems for data mining is important for determining accuracy requirements for iterative methods
- Data Analysis structure can be revealed faster than accurate solutions to numerical problem

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Block Means Are Spread Out

Lemma

Define $\mathcal{B}_i = \{ib + 1 \ldots ib + q\}$
Let $\mathcal{W}$ *be the span of* $\{\mathbf{e}_{\mathcal{B}_i} \mid i \in 0 \ldots q - 1\}$.
For any vector $\mathbf{x}$, *let* $\mu_j = |\mathcal{B}_i|^{-1} \sum_{i \in \mathcal{B}_j} x_i$.
For any vector $\mathbf{x} \in \mathcal{W}$, $\|\boldsymbol{\mu}\|_\infty > n^{-1/2} \|\mathbf{x}\|_2$.

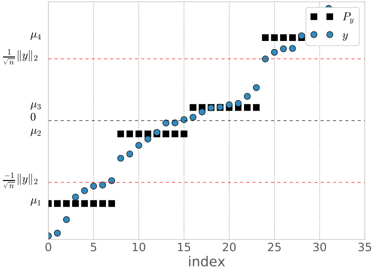

Figure 8: Notation in Lemma